International Journal of Theoretical Physics, Vol. 12, No. 3 (1975), pp. 183–190

Pre-Quantisation for an Arbitrary Abelian Structure Group

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Received: 4 July 1974

Abstract

A geometric study of canonical quantisation generalising the pre-quantisation technique of Kostant is presented. The concept of a quantising fibre bundle with an arbitrary abelian structure group arises naturally in this framework. It is demonstrated that quantising fibre bundles induce vector-field representations. Necessary and sufficient conditions for the quantisability of symplectic manifolds are derived and a proof for the existence of two-dimensional non-reducible quantising toral bundles is given.

1. *Introduction*

The geometric quantisation theories introduced by Segal (1960), Souriau (1967), and Kostant (1970) start with a symplectic manifold (M, ω) and the Lie algebra $\mathfrak{F}(M)$ of smooth functions on M with the Poisson bracket $\{\varphi, \psi\}$. Then geometric pre-quantisation consists of constructing a quantising A-bundle, that means a smooth principal fibre bundle (P, A, M) over (M, ω) with total space P, structure group A, and the additional condition that ω generates a curvature form $\nabla\alpha$ on P. Let C^* be the multiplicative group of non-zero complex numbers. It is known that Kostant's pre-quantisation is characterised by quantising C^*-bundles which are reducible to one-dimensional quantising toral bundles. Similarly, one-dimensional quantising toral bundles arise in Souriau's approach. An essential property of these pre-quantisation procedures is that they induce infinitesimal actions of the Lie algebra $\mathfrak{F}(M)$ on the total space P.

In this context it is natural to ask whether this concept can be extended to arbitrary abelian structure groups and, more important, whether this extension can be made in such a way that the typical pre-quantisation properties are preserved. A treatment of this problem is presented in the quantising fibre bundle notation. We begin by defining quantising fibre bundles for arbitrary abelian structure groups. It is shown that those bundles induce Lie algebra homomorphisms from the Lie algebra $\mathfrak{F}(M)$ into the Lie algebra of A-invariant vector-fields on P. For a detailed characterisation of quantising fibre bundles systems of quantising functions are introduced. This technique is used to discuss the relation between the pre-quantisation procedures of Kostant and Souriau. Moreover, a proof is given for the existence of higher dimensional non-reducible quantising fibre bundles.

2. (A, λ)-*Quantisable Manifolds*

Let (P, A, M) be a smooth principal fibre bundle over a manifold M with abelian structure group A and projection $\pi: P \to M$. Let $x^+ \in \mathfrak{B}(P)$ be the fundamental vector-field corresponding to $x \in \mathfrak{a}$, where $\mathfrak{a}$ denotes the commutative Lie algebra of invariant vector-fields on A. In this case a connection form on P is a 1-form $\alpha \in \mathfrak{A}^1_{\mathfrak{a}}(P)$ with values in $\mathfrak{a}$ such that

(i) $\alpha(x^+) = x$ for each $x \in \mathfrak{a}$;

(ii) α is A-invariant.

The exterior derivative of a V-valued p-form $(p \geqslant 0)$ $\Phi \in \mathfrak{A}^p_V(M)$ will be denoted by $d\Phi$. In particular, $d\alpha$ is the curvature form associated with the connection form α since

$$\nabla\alpha = d\alpha + \tfrac{1}{2}[\alpha, \alpha] = d\alpha$$

Let $\lambda : \mathsf{R} \to \mathfrak{a}$ be a linear injection and (M, ω) be a symplectic manifold, that is, a manifold M together with a closed non-degenerate 2-form ω on M.

Definition 1. (M, ω) is called (A, λ)-quantisable to (P, α) if there exists a principal fibre bundle (P, A, M) over M with a connection form α on P such that $d\alpha = \lambda(\pi^*\omega)$.

Here $\pi^* : \mathfrak{A}^2_{\mathsf{R}}(M) \to \mathfrak{A}^2_{\mathsf{R}}(P)$ denotes the map of the 2-forms on M into the 2-forms on P induced by π, and the map $\mathfrak{A}^2_{\mathsf{R}}(P) \to \mathfrak{A}^2_{\mathfrak{a}}(P)$ associated to $\lambda : \mathsf{R} \to \mathfrak{a}$ will be denoted by λ, too.

We call $(P, \alpha, A, \lambda, M, \omega)$ a quantising fibre bundle if (M, ω) is (A, λ)-quantisable to (P, α). Examples will be considered in Section 6.

3. *Vector-field Representations*

Given a quantising fibre bundle $(P, \alpha, A, \lambda, M, \omega)$, we denote by $\mathfrak{B}_A(P)$ the Lie algebra of A-invariant vector-fields on P. Then

$$\mathfrak{B}_A(P) = \mathfrak{B}^V_A(P) + \mathfrak{B}^H_A(P)$$

where $\mathfrak{B}^V_A(P)$ denotes the linear space of vertical and $\mathfrak{B}^H_A(P)$ denotes the linear space of horizontal A-invariant vector-fields on P with respect to the connection form α. Let $\tilde{\xi} \in \mathfrak{B}^H_A(P)$ be the horizontal lift of a vector-field $\xi \in \mathfrak{B}(M)$. Then the map

$$\eta : \mathfrak{F}_{\mathfrak{a}}(M) + \mathfrak{B}(M) \to \mathfrak{B}_A(P)$$

defined by

$$\eta(\Phi, \xi) \,|\, \pi^{-1}(m) = -(\Phi(m))^{+} \,|\, \pi^{-1}(m) + \tilde{\xi} \,|\, \pi^{-1}(m)$$

$\Phi \in \mathfrak{F}_{\mathfrak{a}}(M), m \in M$, is a linear isomorphism.

Now consider a symplectic manifold (M, ω); it is known that the symplectic structure defines an isomorphism $\mathfrak{V}(M) \to \mathfrak{A}^1(M)$ by $\xi \to \beta_\xi = i_\xi \omega$, where i_ξ is the substitution operator induced by $\xi \in \mathfrak{V}(M)$. Moreover, ω determines a map $\varphi \in \mathfrak{F}(M) \to \xi_\varphi \in \mathfrak{V}(M)$ by $\beta_{\xi_\varphi} = d\varphi$. It can be shown that

$$\{\varphi, \psi\} = \xi_\varphi \psi = \omega(\xi_\psi, \xi_\varphi)$$

$\varphi, \psi \in \mathfrak{F}(M)$, is a Lie algebra structure on $\mathfrak{F}(M)$ such that $\varphi \to \xi_\varphi$ becomes a Lie algebra homomorphism. The following proposition is a straightforward generalisation of a result given by Kostant (1970).

Proposition 1. Let $(P, \alpha, A, \lambda, M, \omega)$ be a quantising fibre bundle. Then for $\varphi_i \in \mathfrak{F}(M), i = 1, 2$

$$[\eta(\lambda\varphi_1, \xi_{\varphi_1}), \eta(\lambda\varphi_2, \xi_{\varphi_2})] = \eta(\lambda\{\varphi_1, \varphi_2\}, [\xi_{\varphi_1}, \xi_{\varphi_2}])$$

Since $\varphi \to \xi_\varphi$ is a Lie algebra homomorphism, Proposition 1 gives the following infinitesimal action of $\mathfrak{F}(M)$ on P:

Theorem 1. Let $(P, \alpha, A, \lambda, M, \omega)$ be a quantising fibre bundle. Then the map

$$\delta : \mathfrak{F}(M) \to \mathfrak{V}_A(P)$$

defined by $\delta(\varphi) = \eta(\lambda\varphi, \xi_\varphi)$ for $\varphi \in \mathfrak{F}(M)$ is an injective Lie algebra homomorphism.

Let $\rho : A \to \mathrm{Aut}\, V$ be a finite-dimensional linear representation of A. Denote by $\mathfrak{F}^{\rho}_{V}(P)$ the space of smooth maps $f: P \to V$ such that $f(pa) = \rho(a^{-1})f(p)$ for $p \in P, a \in A$. An easy calculation shows that $\mathfrak{F}^{\rho}_{V}(P)$ is an invariant domain for the linear operators $\delta(\varphi), \varphi \in \mathfrak{F}(M)$. It follows that

$$\delta^{\rho} : \mathfrak{F}(M) \to \mathrm{End}\ \mathfrak{F}^{\rho}_{V}(P)$$

is a linear representation of $\mathfrak{F}(M)$ by vector-field operators. Thus associated to each quantising fibre bundle $(P, \alpha, A, \lambda, M, \omega)$ are vector-field representations of $\mathfrak{F}(M)$ on $\mathfrak{F}^{\rho}_{V}(P)$.

4. *Systems of Quantising Functions*

We introduce systems of quantising functions and establish some of their elementary properties. Applications will be given in the next two sections.

Let $\mathfrak{U} = \{U_i; i \in I\}$ be an open covering of a symplectic manifold (M, ω) and let A be an abelian Lie group with Lie algebra $\mathfrak{a}$.

Definition 2. A system $\{f_{ij}, \alpha_i; i, j \in I\}$ of $(A, \lambda, \mathfrak{U}, \omega)$-quantising functions consists of transition functions $f_{ij} \in \mathfrak{F}_A(U_i \cap U_j)$ and $\mathfrak{a}$-valued 1-forms $\alpha_i \in \mathfrak{A}^1_{\mathfrak{a}}(U_i)$ such that for $i, j \in I$

(i) there are $\alpha_{ij} \in \mathfrak{F}_{\mathfrak{a}}(U_i \cap U_j)$ with

$$f_{ij} = \exp \alpha_{ij}, \qquad d\alpha_{ij} = \alpha_j - \alpha_i$$

on $U_i \cap U_j$;

(ii) $d\alpha_i = \lambda(\omega)$ on U_i.

Here $\exp : \mathfrak{a} \to A$ denotes the exponential map.

A system of $(A, \lambda, \mathfrak{U}, \omega)$-quantising functions is said to be q-dimensional if $\dim A = q$. Two systems $\{f_{ij}, \alpha_i; i, j \in I\}$ and $\{f'_{ij}, \alpha'_i; i, j \in I\}$ of $(A, \lambda, \mathfrak{U}, \omega)$-quantising functions are called equivalent if the corresponding transition functions are equivalent (i.e. $f'_{ij} = g_i^{-1} f_{ij} g_j$ for suitable $g_i \in \mathfrak{F}_A(U_i)$) and if there exist $\beta_i \in \mathfrak{F}_{\mathfrak{a}}(U_i)$ such that

(i) $\alpha'_i = \alpha_i + d\beta_i$;

(ii) $g_i = \exp \beta_i$.

Let A' be a Lie subgroup of A. A system $\{f_{ij}, \alpha_i; i, j \in I\}$ of $(A, \lambda, \mathfrak{U}, \omega)$-quantising functions is called A'-reducible if the f_{ij} take their values in A' and the α_i and λ take their values in $\mathfrak{a}'$. It is easy to check that an A'-reducible system $\{f_{ij}, \alpha_i; i, j \in I\}$ of $(A, \lambda, \mathfrak{U}, \omega)$-quantising functions defines a system $\{f'_{ij}, \alpha'_i; i, j \in I\}$ of $(A', \lambda', \mathfrak{U}, \omega)$-quantising functions in a natural way. Conversely, any system $\{f'_{ij}, \alpha'_i; i, j \in I\}$ of $(A', \lambda', \mathfrak{U}, \omega)$-quantising functions induces an A'-reducible system $\{f_{ij}, \alpha_i; i, j \in I\}$ of $(A, \lambda, \mathfrak{U}, \omega)$-quantising functions.

Let $\{f_{ij}, \alpha_i; i, j \in I\}$ be a system of $(A, \lambda, \mathfrak{U}, \omega)$-quantising functions. Then, for $a \in \mathrm{R}, a \neq 0$, $\{f_{ij}, \alpha_i; i, j \in I\}$ can be considered as a system of $(A, a^{-1}\lambda, \mathfrak{U}, a\omega)$-quantising functions. In other words, if (M, ω) is (A, λ)-quantisable, then $(M, a\omega)$ is $(A, a^{-1}\lambda)$-quantisable for $a \in \mathrm{R}, a \neq 0$.

5. *Characterisation of Quantising Fibre Bundles*

Let $Q_i = (P_i, \alpha_i, A_i, \lambda_i, M_i, \omega_i)$ $(i = 1, 2)$ be two quantising fibre bundles. A quantising fibre bundle homomorphism $f : Q_1 \to Q_2$ is a principal fibre bundle homomorphism $(f', f'') : (P_1, A_1, M_1) \to (P_2, A_2, M_2)$ such that

(i) $f''_* \lambda_1 = \lambda_2$;

(ii) $f'^* \alpha_2 = f''_* \alpha_1$.

Q_1 and Q_2 are called equivalent if there exists a homomorphism $f : Q_1 \to Q_2$ which is an equivalence of principal fibre bundles. Let A_1 be a Lie subgroup of A_2. Q_2 is said to be A_1-reducible if there is a homomorphism $f : Q_1 \to Q_2$ which is a reduction of the structure group A_2 of (P_2, A_2, M_2) to A_1.

A trivialisation $\{U_i, \varphi_i; i \in I\}$ of a principal fibre bundle is called simply connected if the intersection of a finite family of $U_i, i \in I$, is simply connected; it is well known that for every open covering of the base space there is a simply connected open refinement.

Principal fibre bundles are represented by systems of transition functions. This method can be extended to quantising fibre bundles.

Theorem 2. (1) Let $(P, \alpha, A, \lambda, M, \omega)$ be a quantising fibre bundle and $\{U_i, \varphi_i; i \in I\}$ a simply connected trivialisation of (P, A, M). Then there exists an associated system $\{f_{ij}, \alpha_i; i, j \in I\}$ of $(A, \lambda, \mathfrak{U}, \omega)$-quantising functions with $\mathfrak{U} = \{U_i; i \in I\}$. Conversely, given a system $\{f_{ij}, \alpha_i; i, j \in I\}$ of $(A, \lambda, \mathfrak{A}, \omega)$-quantising functions, then there exists an associated quantising fibre bundle $(P, \alpha, A, \lambda, M, \omega)$ having $\{f_{ij}, \alpha_i; i, j \in I\}$ as associated system of quantising functions.

(2) Two quantising fibre bundles $(P, \alpha, A, \lambda, M, \omega)$ and $(P', \alpha', A, \lambda, M, \omega)$ are equivalent if and only if there exist simply connected trivialisations $\{U_i, \varphi_i; i \in I\}$ and $\{U_i, \varphi_i'; i \in I\}$ such that the associated systems $\{f_{ij}, \alpha_i; i, j \in I\}$ and $\{f_{ij}', \alpha_i'; i, j \in I\}$ of $(A, \lambda, \mathfrak{U}, \omega)$-quantising functions are equivalent.

(3) Let A' be a Lie subgroup of A. A quantising fibre bundle $(P, \alpha, A, \lambda, M, \omega)$ is A'-reducible if and only if there exists an associated A'-reducible system $\{f_{ij}, \alpha_i; i, j \in I\}$ of $(A, \lambda, \mathfrak{U}, \omega)$-quantising functions.

Before proving Theorem 2 we establish three propositions. Define an $\mathfrak{a}$-valued 1-form $\theta \in \mathfrak{A}^1_{\mathfrak{a}}(A)$ on A by $\theta_a(x_a) = x$ for $x \in \mathfrak{a}, a \in A$. Then we have

Proposition 2. Let (P, A, M) be a principal fibre bundle with abelian structure group A and transition functions $f_{ij} \in \mathfrak{F}_A(U_i \cap U_j), i, j \in I$. If there are $\alpha_{ij} \in \mathfrak{F}_{\mathfrak{a}}(U_i \cap U_j)$ with $f_{ij} = \exp \alpha_{ij}$, then for $i, j \in I$

$$d\alpha_{ij} = f_{ij}^* \theta$$

Proof. For elements ξ_x of the tangent space $T_x(U_i \cap U_j)$ at $x \in U_i \cap U_j$ we obtain

$$\begin{aligned}(f_{ij}^*\theta)(\xi_x) &= \theta((f_{ij})_*\xi_x) = (f_{ij})_*\xi_x \\ &= \exp_*(\alpha_{ij})_*\xi_x = d\alpha_{ij}(\xi_x)\end{aligned}$$

Now consider a principal fibre bundle (P, A, M) with trivialisation $\{U_i, \varphi_i; i \in I\}$ and associated system $\{f_{ij}; i, j \in I\}$ of transition functions. For each $i \in I$ let $\sigma_i : U_i \to \pi^{-1}(U_i)$ be the cross section defined by $\sigma_i(x) = \varphi_i(x, e)$, $x \in U_i$.

Proposition 3. Let (P, A, M) be a principal fibre bundle with abelian structure group A and transition functions $f_{ij} \in \mathfrak{F}_A(U_i \cap U_j), i, j \in I$. Given a connection form α on (P, A, M), the 1-forms $f_{ij}^*\theta$ and $\alpha_i = \sigma_i^*\alpha$ satisfy the conditions

$$\alpha_j = \alpha_i + f_{ij}^*\theta$$

on $U_i \cap U_j, i, j \in I$. Conversely, given a family $\alpha_i \in \mathfrak{A}^1_{\mathfrak{a}}(U_i)$ satisfying the preceding conditions, we can construct a unique connection form α on P with $\alpha_i = \sigma_i^*\alpha$.

For the proof of Proposition 3 see Kobayashi & Nomizu (1963), p. 66.

In terms of φ_i each $\xi \in \mathfrak{B}(\pi^{-1}(U_i))$ may be expressed by

$$\xi_{(x,a)} = \xi^1_{(x,a)} + \xi^2_{(x,a)}$$

for $(x, a) \in U_i \times A$, $\xi^1_{(x,a)} \in T_x(U_i)$, $\xi^2_{(x,a)} \in T_a(A)$.

Proposition 4. With the notation above,

$$d\alpha \,|\, \pi^{-1}(U_i) = \pi^* \, d\alpha_i$$

Proof. Consider $\xi \in \mathfrak{B}(U_i) \cup \mathfrak{a}$, $\eta \in \mathfrak{a}$. ξ, η can be regarded as elements of $\mathfrak{B}(U_i \times A)$. Since α and ξ are A-invariant, $\alpha(\xi) | x \times A$ is constant for $x \in U_i$. Clearly, $\alpha(\eta) = \eta$ is constant on $U_i \times A$. Consequently

$$d\alpha(\xi_{(x,a)}, \eta_{(x,a)}) = \xi_{(x,a)}\alpha(\eta) - \eta_{(x,a)}\alpha(\xi) - \alpha_{(x,a)}([\xi, \eta]) = 0$$

and we obtain

$$d\alpha(\xi^1_{(x,a)} + \xi^2_{(x,a)}, \eta^1_{(x,a)} + \eta^2_{(x,a)}) = d\alpha(\xi^1_{(x,a)}, \eta^1_{(x,a)})$$
$$= (\pi^* \, d\alpha_i)(\xi^1_{(x,a)}, \eta^1_{(x,a)}) = (\pi^* \, d\alpha_i)(\xi^1_{(x,a)} + \xi^2_{(x,a)}, \eta^1_{(x,a)} + \eta^2_{(x,a)})$$

Proof of Theorem 1

(1) Given a quantising fibre bundle $(P, \alpha, A, \lambda, M, \omega)$ with a simply connected trivialisation $\{U_i, \varphi_i; i \in I\}$ of (P, A, M), consider the associated system $\{f_{ij}; i, j \in I\}$ of transition functions. Since the $U_i \cap U_j$ are simply connected, there are $\alpha_{ij} \in \mathfrak{F}_{\mathfrak{a}}(U_i \cap U_j)$ with $f_{ij} = \exp \alpha_{ij}$. Now Proposition 2 and Proposition 3 give $\alpha_j - \alpha_i = d\alpha_{ij}$ for $\alpha_i = \sigma_i^* \alpha$. According to Definition 1 we have

$$d\alpha_i = \sigma_i^* \, d\alpha = \lambda(\sigma_i^* \pi^* \omega) = \lambda(\omega) \,|\, U_i$$

Consequently $\{f_{ij}, \alpha_i; i, j \in I\}$ is a system of $(A, \lambda, \mathfrak{U}, \omega)$-quantising functions.

Conversely, given a system $\{f_{ij}, \alpha_i; i, j \in I\}$ of $(A, \lambda, \mathfrak{U}, \omega)$-quantising functions, consider the principal fibre bundle (P, A, M) associated to $\{f_{ij}; i, j \in I\}$. Then it follows from Proposition 2 and Proposition 3 that there is a unique connection form α on P such that $\alpha_i = \sigma_i^* \alpha$. As an immediate consequence of Proposition 4 we obtain for $i \in I$

$$d\alpha \,|\, \pi^{-1}(U_i) = \pi^* \, d\alpha_i = \pi^* \lambda(\omega \,|\, U_i)$$

i.e. $(P, \alpha, A, \lambda, M, \omega)$ is a quantising fibre bundle.

It is easily checked that $\{f_{ij}, \alpha_i; i, j \in I\}$ is the system of quantising functions associated to $(P, \alpha, A, \lambda, M, \omega)$ and $\{U_i, \varphi_i; i \in I\}$.

(2) By a standard theorem in the theory of fibre bundles an equivalence $\Phi : (P, A, M) \to (P', A, M)$ is characterised by functions $g_i \in \mathfrak{F}_A(U_i)$ such that for $i, j \in I$

(i) $\Phi(\varphi_i(x, a)) = \varphi_i'(x, g_i^{-1}(x)a)$;

(ii) $f_{ij}' = g_i^{-1} f_{ij} g_j$.

Here $\{U_i, \varphi_i; i \in I\}$ and $\{U_i, \varphi_i'; i \in I\}$ are trivialisations of (P, A, M) and (P', A, M) with associated transition functions f_{ij} and f_{ij}', respectively. We may assume that the trivialisations are simply connected (by taking a refinement if necessary). Hence there are $\beta_i \in \mathfrak{F}_{\mathfrak{a}}(U_i)$ with $\exp \beta_i = g_i$ for $i \in I$. It remains to prove that

$$\alpha = \Phi^* \alpha' \text{ if and only if } \alpha_i' = \alpha_i + d\beta_i$$

for $\alpha_i = \sigma_i^* \alpha$, $\alpha_i' = \sigma_i'^* \alpha'$, $i \in I$. An easy calculation gives

(a) $\Phi_* \xi_{(x,a)} = \xi^1_{(x,a)} + (g_i^{-1})_* \xi^1_{(x,a)} + \xi^2_{(x,a)}$,

(b) $\alpha(\xi_{(x,a)}) = \alpha_i(\xi^1_{(x,a)}) + \xi^2_{(x,a)}$,

(c) $d\beta_i(\xi^1_{(x,a)}) = -(g_i^{-1})_* \xi^1_{(x,a)}$

for $\xi_{(x,a)} \in T_{(x,a)}(U_i \times A)$, $\xi_{(x,a)} = \xi^1_{(x,a)} + \xi^2_{(x,a)}$, $i \in I$. Hence $\alpha = \Phi^*\alpha'$ if and only if

$$\alpha_i(\xi^1_{(x,a)}) = \alpha_i'(\xi^1_{(x,a)}) + (g_i^{-1})_* \xi^1_{(x,a)}$$

But according to (c) this is equivalent to $\alpha_i' = \alpha_i + d\beta_i$ for $i \in I$.

(3) It is known that a reduction $\Phi : (P', A', M) \to (P, A, M)$ is characterised by the existence of trivialisations $\{U_i, \varphi_i'; i \in I\}$ and $\{U_i, \varphi_i; i \in I\}$ of (P', A', M) and (P, A, M) such that

(i) $\Phi(\varphi_i'(x, a')) = \varphi_i(x, a')$ for $(x, a') \in U_i \times A'$;

(ii) the transition functions f_{ij} associated to $\{U_i, \varphi_i; i \in I\}$ take their values in A'.

Given two connection forms α' and α on P', P respectively, it remains to show that

$$\Phi^*\alpha = \alpha' \text{ if and only if } \alpha_i(\xi^1_{(x,a)}) = \alpha_i'(\xi^1_{(x,a)})$$

for $\xi^1_{(x,a)} \in T_x(U_i)$, $x \in U_i$, $i \in I$. Since

$$\Phi^*\alpha = \alpha' \text{ if and only if } \alpha(\xi_{(x,a')}) = \alpha'(\xi_{(x,a')})$$

for $\xi_{(x,a')} \in T_{(x,a')}(U_i \times A')$, the proof follows immediately from (b) in (2).

6. *Comparison with the Theories of Kostant and Souriau*

It is well known (Hurt, 1968) that an *espace fibré quantifiant* (Souriau, 1967) is given by a principal toral bundle (P, T, M) with connection form α over a symplectic manifold (M, ω) such that $\pi^*\omega = d\alpha$. In other words, an *espace fibré quantifiant* may be regarded as a quantising fibre bundle with structure group $\mathsf{T} = \mathsf{R}/\mathsf{Z}$ and injection $\lambda = id_{\mathsf{R}}$. Note that in this case $\exp : \mathsf{R} \to \mathsf{T}$ is defined by $\exp(x) = e^{ix}$ for $x \in \mathsf{R}$.

On the other hand, the integrality condition given by Kostant (1970) is a necessary and sufficient condition for the existence of T-reducible systems of $(\mathsf{C}^*, \lambda_{2\pi}, \mathfrak{U}, \omega)$-quantising functions. Here $\lambda_{2\pi} : \mathsf{R} \to \mathsf{C}$ is defined by $\lambda_{2\pi}(x) = 2\pi x$, $x \in \mathsf{R}$, and $\exp : \mathsf{C} \to \mathsf{C}^* = GL(1, \mathsf{C})$ is given by $\exp(z) = e^{iz}$ for $z \in \mathsf{C}$. Moreover, Kostant's Hermitian line bundles over a symplectic manifold can be identified with T-reducible quantising fibre bundles with structure group C^* and injection $\lambda_{2\pi}$. The natural action of C^* on C yields a representation $\rho : \mathsf{C}^* \to \text{Aut } \mathsf{C}$. In this case the vector-field representation δ^ρ is equivalent to Kostant's pre-quantisation.

A symplectic manifold (M, ω) is $(\mathsf{T}, id_{\mathsf{R}})$-quantisable if and only if $(M, \omega/2\pi)$ is $(\mathsf{C}^*, \lambda_{2\pi})$-quantisable to a T-reducible quantising fibre bundle. Thus the equivalence classes of *espaces fibrés quantifiants* are in a natural one–one correspondence with the equivalence classes of Kostant's T-reducible quantising C^*-bundles.

Another class of quantising fibre bundles can be constructed by changing the structure group. One knows that there are non-equivalent systems of $(\mathsf{T}, id_{\mathsf{R}}, \mathfrak{U}, \omega)$-quantising functions. Thus the existence of non-reducible two-dimensional quantising toral bundles follows from the next result.

Proposition 5. Given two systems $\{f_{ij}^{\nu}, \alpha_i^{\nu}; i, j \in I\}$ $(\nu = 1, 2)$ of $(\mathsf{T}, id_{\mathsf{R}}, \mathfrak{U}, \omega)$-quantising functions, then

$$\{f_{ij} = (f_{ij}^1, f_{ij}^2), \alpha_i = \alpha_i^1 \oplus \alpha_i^2; i, j \in I\}$$

is a system of $(\mathsf{T} \times \mathsf{T}, id_{\mathsf{R}} \oplus id_{\mathsf{R}}, \mathfrak{U}, \omega)$-quantising functions. Moreover, if $\{f_{ij}, \alpha_i; i, j \in I\}$ is equivalent to a T-reducible system $\{f_{ij}', \alpha_i'; i, j \in I\}$, then the two systems $\{f_{ij}^{\nu}, \alpha_i^{\nu}; i, j \in I\}$ are equivalent.

Proof. The first part is easily checked. For the second part, consider $\beta_i \in \mathfrak{F}_{\mathsf{R}^2}(U_i)$ such that for $i \in I$

(i) $\exp \alpha_{ij}' = \exp(-\beta_i + \alpha_{ij} + \beta_j)$;

(ii) $\alpha_i' = \alpha_i + d\beta_i$.

The reducibility of $\{f_{ij}', \alpha_i'; i, j \in I\}$ tells us that

$$\exp \alpha_{ij}' = \exp(a_{ij}', a_{ij}'), \qquad \alpha_i' = (a_i', a_i')$$

for suitable $a_{ij}' \in \mathfrak{F}(U_i)$, $a_i' \in \mathfrak{A}^1(U_i)$. Then the first thing to notice is that (i) implies

$$\exp(a_{ij}', a_{ij}') = \exp(-\beta_i^1 + \alpha_{ij}^1 + \beta_j^1, -\beta_i^2 + \alpha_{ij}^2 + \beta_j^2)$$

for $\beta_i = (\beta_i^1, \beta_i^2)$, $\beta_i^1, \beta_i^2 \in \mathfrak{F}(U_i)$, $i \in I$. Consequently

$$\exp \alpha_{ij}^1 = \exp(-(\beta_i^2 - \beta_i^1) + \alpha_{ij}^2 + (\beta_j^2 - \beta_j^1)) \tag{1}$$

On the other hand

$$\alpha_i + d\beta_i = (\alpha_i^1 + d\beta_i^1, \alpha_i^2 + d\beta_i^2)$$

Hence

$$\alpha_i^1 = \alpha_i^2 + d(\beta_i^2 - \beta_i^1) \tag{2}$$

(1) and (2) prove the assertion.

Acknowledgement

The author is indebted to Prof. Dr. H. D. Doebner for drawing his attention to this problem, and gives thanks to Dr. J. Hennig and Dipl. Phys. F. B. Pasemann for helpful discussions.

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